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Abstract

We consider the question of optimizing the separative power of a centrifuge and we treat it as an optimal control problem.

We first explain how this problem may be embedded in the framework of optimal Control Theory. The state of the distributed system is given by the linearized system of equations of motion, the cost function is a functional expressing the separative power of the centrifuge and we take boundary controls representing the temperature field imposed on the boundary of the centrifuge.

Then, applying some variants of classical gradient and conjugate gradient methods, we obtain numerical values for the optimal control, that is for the temperature field maximizing the separative power of the centrifuge (we also present various verifications of these results). These numerical results seem to indicate some interesting and unexpected phenomena concerning the gas flow in the centrifuge.

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Introduction

The separative performance of a gas centrifuge used for uranium enrichment [1] depends strongly on the gas flow field inside the device. This flow is generally activated by boundary conditions of two types:

- 1 - Temperature distribution on the solid boundary, i.e the two end plates and the side wall, assumed to be perfectly conducting. This yields the "thermal drive".
 - 2 - Angular momentum sink distribution near one end plate, simulating quite roughly a scoop which generates the "friction drive".
- The problem to be solved is: how to select the boundary conditions to get the highest separative performance of a centrifuge?
 - A simplified approach of this question has been developed in [2]. The present paper proceeds to an extension of that previous work by embedding the problem in the framework of the general theory of OPTIMAL CONTROL [3].

In Sec I we recall first a few basic notions of the theory of optimal control of distributed parameter systems and we then explain how we can apply these notions to the problem of ultracentrifugation considered here. The state of the distributed system is governed by the linearized system of hydrodynamic equations. The cost function is a functional expressing the separative power of the centrifuge. The control is of the type "boundary control" and represents the temperature distribution on the solid boundaries. We restrict the analysis to the thermal drive because it was pointed out by many authors (see for instance [4] that the friction

drive, simulated by angular momentum sink at one end plate, is approximately the same as an end cap thermal drive.

In Sec II we deal with the numerical method built up to solve this particular optimal control problem. The method is in fact a variant of classical gradient and conjugate gradient methods. The solution requires the use of the CENTAURE code [5].

Finally in Sec III we present the numerical results of a sample computation carried out for one centrifuge chosen to illustrate the method and we discuss the results.

I. Mathematical Description of the Problem

I. 1. Some basic notions of optimal control of distributed parameters systems

We describe briefly the general framework of optimal control of distributed parameter systems and we will consider the case of boundary controls.

We consider a system which state y is defined by the equation:

$$(1) \quad \begin{cases} \mathcal{A}_y = 0 \\ \mathcal{B}_y = v \end{cases},$$

where \mathcal{A} denotes a linear partial differential operator defined on a functional Hilbert space X (space of functions defined on a bounded open set Ω of \mathbb{R}^N) and where \mathcal{B} denotes a "boundary operator" that is a linear operator from X into a Hilbert space Y consisting of functions defined on $\partial\Omega$ (the boundary of Ω). Finally v is an ar-

bitrary element of Y and will be the control.

We assume that the problem (1) is well-posed that is for each $v \in Y$, there exists a unique solution $y \in X$: of course y depends linearly of v and we will sometimes write $y=y(v)$. Without loss of generality, we may assume that we have certain constraints on v : we require v to be in a set \mathcal{U}_{ad} and we assume:

(2) \mathcal{U}_{ad} is a closed, convex set of Y .

Let \tilde{J} be a C^1 functional over X , we define for each $v \in \mathcal{U}_{ad}$ a cost function $J(v)$ by

$$(3) \quad J(v) = \tilde{J}(y(v)).$$

The optimal control problem is to minimize J over \mathcal{U}_{ad} and to find the optimal control v^* that is to find v^* satisfying

$$(4) \quad J(v^*) = \inf_{v \in \mathcal{U}_{ad}} J(v), \quad v^* \in \mathcal{U}_{ad}.$$

In the application we have in mind J is not convex and a typical assumption which insures the existence of v^* satisfying (4) is that \mathcal{U}_{ad} is bounded and that \tilde{J} is weakly lower-semi-continuous - for more details the reader is referred to J.L. LIONS [3], D.L. RUSSEL [6] and to [7].

Now, to compute such a v^* , a simple (and robust) algorithm is the so-called projected gradient method: let $v^* \in \mathcal{U}_{ad}$ we define recursively $(p^n, v^n) \in \mathbb{R} \times \mathcal{U}_{ad}$ as follows:

$$(5) \quad \rho^n \text{ minimizes: } J(v^n - \rho^n J'(v^n)) = \inf_{\rho \geq 0} J(v^n - \rho J'(v^n)).$$

$$(6) \quad v^{n+1} = P_{u_{ad}} [v^n - \rho^n J'(v^n)],$$

where $P_{u_{ad}}$ denotes the usual euclidean projection in u_{ad} .

Let us make the following trivial observation (which will be used in the following): if for all $n \geq 0$, the above algorithm generates a sequence v^n such that $v^n - \rho^n J'(v^n) \in u_{ad}$ for all $n \geq 0$ then obviously (6) reduces to

$$(6') \quad v^{n+1} = v^n - \rho^n J'(v^n)$$

which is of course the usual gradient method.

Finally let us mention that many variants of the above algorithms exist (and we will use one of these) - see for example POLAK [8] and J. CEA [9], and we refer to [9] for various results concerning the convergence of the above algorithms.

1.2. Description of a problem of ultracentrifugation

We will follow the approach and model of SOUBBARAMAYER [1], [2], [5]. Our problem takes place in a cylinder (the centrifuge) in $\mathbb{R}^2: \Omega = (0, H) \times (0, 1)$, where H denotes the scaled height of the cylinder (height over radius) and ξ is the radial coordinate lying in $(0, 1)$.

The state of the system is given by the set of hydrodynamic equations linearized around the equilibrium flow (see for more details [1]): $y = (\hat{u}, \hat{v}, \hat{w}, \hat{p}, \hat{T})$ is given by the following equations in

$$(7) \quad \begin{cases} \left(\frac{\xi}{E_R}\right) \left\{ \left(\tilde{\Delta} - \frac{1}{\xi^2}\right) \hat{u} + \frac{1}{3} \frac{\partial}{\partial \xi} \operatorname{div} q \right\} = -2\hat{v} + \xi \hat{T} + \frac{1}{2A^2} \frac{\partial \hat{p}}{\partial \xi} \text{ in } \Omega, \\ \left(\frac{\xi}{E_R}\right) \left\{ \left(\tilde{\Delta} - \frac{1}{\xi^2}\right) \hat{v} \right\} = \hat{u} \text{ in } \Omega, \\ \left(\frac{\xi}{E_R}\right) \left\{ \tilde{\Delta} \hat{w} + \frac{1}{3} \frac{\partial}{\partial \eta} \operatorname{div} q \right\} = \frac{1}{2A^2} \frac{\partial \hat{p}}{\partial \eta} \text{ in } \Omega, \\ \left(\frac{\xi}{E_R}\right) \tilde{\Delta} \hat{T} = -L_1 b \xi \hat{u} \text{ in } \Omega, \\ \operatorname{div} q + 2A^2 \xi \hat{u} = 0 \text{ in } \Omega, \end{cases}$$

where $(\hat{u}, \hat{v}, \hat{w})$ corresponds to the velocity field, \hat{p} to the pressure and \hat{T} to the temperature, and where we used the following notations:

$$\tilde{\Delta} = \frac{\partial^2}{\partial \xi^2} + \frac{1}{\xi} \frac{\partial}{\partial \xi} + \frac{\partial^2}{\partial \eta^2},$$

$$\operatorname{div} q = \frac{1}{\xi} \frac{\xi}{\partial \xi} (\xi \hat{u}) + \frac{\partial \hat{w}}{\partial \eta}.$$

Before presenting the boundary values (where the control will come in), let us mention that ξ is the Ekman number at the periphery and that E_R and b are given by

$$E_R = \exp[-A^2(1 - \xi^2)],$$

$$b = (\gamma - 1) \operatorname{Pr} A^2,$$

where γ and Pr are two dimensionless numbers depending only on the gas used and where A is a dimensionless constant depending on the gas used, the average temperature and the peripheral speed of the cylinder.

We now explain briefly what are the boundary conditions:

$$\text{if } \xi = 0, \hat{u} = \hat{v} = 0, \frac{\partial \hat{w}}{\partial \xi} = \frac{\partial \hat{T}}{\partial \xi} = 0;$$

if $\eta = 0$, $\hat{u} = \hat{w} = 0$, $\hat{v} = \tau(\xi)$, $\hat{T} = v_1(\xi)$;

if $\eta = H$, $\hat{u} = \hat{v} = \hat{w} = 0$, $\hat{T} = v_3(\xi)$;

if $\xi = 1$, $\hat{u} = \hat{v} = \hat{w} = 0$, $\hat{T} = v_2(\eta)$,

where ϕ_1 is some prescribed function of ξ . Finally we define the control v as the collection of $v_1(\xi)$, $v_2(\eta)$, $v_3(\xi)$ or more simple as a function on $\partial_0 \Omega =$

$\{(\xi, 0), \xi \in [0, 1]\} \cup \{(1, \eta), \eta \in [0, H]\} \cup \{(\xi, H), \xi \in [0, 1]\}$

and we have $\hat{T} = v$ on $\partial_0 \Omega$.

Finally let us mention that this model is valid as long as we have

$$(9) \quad \max_{\partial_0 \Omega} |V| \leq \delta_0$$

and δ_0 is some given constant, so $\mathcal{U}_{ad} = \{v, \max_{\partial_0 \Omega} |v| \leq \delta_0\}$.

As we do not want to enter unuseful technicalities, we will not make precise in which functional spaces the boundary value problem (7) - (8) is well-posed and in which space we require the control to belong.

We now define the cost function: following [2], we introduce a few notations, we define the stream function

$$\psi(\eta, \xi) = \int_0^\xi E_R \hat{w} \xi' d\xi'$$

and we introduce the following functions:

$$J_1(\eta) = 2S_c \xi^{-1} \int_0^1 \psi \xi d\xi,$$

$$J_2(\eta) = 2S_c^2 \xi^{-1} \int_0^1 \psi^2 \frac{d\xi}{\xi},$$

$$F(\eta) = \int_{\eta_F}^\eta \frac{\phi_p + 2\xi_0 J_1(\eta')}{1 + J_2(\eta')} d\eta',$$

$$G(\eta) = \int_0^\eta \frac{-\phi_w + 2\xi_0 J_1(\eta')}{1 + J_2(\eta')} d\eta',$$

where S_c , ϕ_p , ϕ_w , η_F , ξ_0 are some given constants (independent of the state of the system).

The separative power, that we wish to maximize is given by the following formula (see [1]):

$$\delta L = F_X \{-\theta \text{Log} \alpha_S + \text{Log} (1 + (\alpha_S - 1)\theta)\},$$

where α_S is given by

$$\alpha_S = \frac{\exp \{G(\eta_F)\} \{1 + \phi_w \int_0^{\eta_F} \frac{\exp[-G(\eta)]}{1 + J_2(\eta)} d\eta\}}{\exp [-F(H)] + \phi_p \int_{\eta_F}^H \frac{\exp[-F(\eta)]}{1 + J_2(\eta)} d\eta}$$

and F and θ are prescribed constant.

We have thus define a cost function for each state y

$$\tilde{J}(y) = -\delta L$$

and we want to minimize

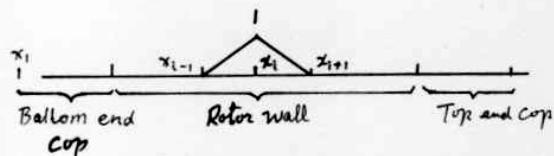
$$\inf_{v \in \mathcal{U}_{ad}} \{J(v) = \tilde{J}(y(v))\},$$

where $\mathcal{U}_{ad} = \{v, \max_{\partial_0 \Omega} |v| \leq \delta_0\}$.

Numerical Method

1. - Approximation space.

Let us define X_h the piecewise linear, continuous functions on the boundary $\partial\Omega$; we search for $v \in X_h$. We assume $(x_i)_{i=1}^N$ is a partition of $\partial\Omega$, and $(v_i)_{i=1}^N$ are N functions of X_h defined by: $v_i(x_j) = 0$ where $j \neq i$ and $v_i(x_i) = s$.



Therefore the search for $v \in X_h$ is equivalent to the search for $(\lambda_i)_{i=1}^N \in \mathbb{R}^N$ such that $v = \sum_{i=1}^N \lambda_i v_i$. Then, the

approximate problem is to find $(\lambda_i^*)_{i=1}^N \in \mathbb{R}^N$ satisfying:

$$J\left(\sum_{i=1}^N \lambda_i^* v_i\right) = \inf_{(\lambda_i)_{i=1}^N \in \mathbb{R}^N} J\left(\sum_{i=1}^N \lambda_i v_i\right)$$

$$v = \sum_{i=1}^N \lambda_i v_i \in \mathcal{U}_{ad}$$

Remark: $v = \sum_{i=1}^N \lambda_i v_i \in \mathcal{U}_{ad}$ is equivalent to $\max_{i=1, N} |\lambda_i| \leq \partial_0$.

Notation: for a fixed base $(v_i)_{i=1}^N$ we write $J((\lambda_i)_{i=1}^N) = J\left(\sum_{i=1}^N \lambda_i v_i\right)$. Let $\vec{v} = (\lambda_i)_{i=1}^N \in \mathbb{R}^N$ and $\vec{W} = \vec{v} J((\lambda_i)_{i=1}^N)$; to find

* minimising $J(\vec{v} - \rho \vec{W})$, we use the "dichotomic method with parabolic fit".

Let us fix $\rho > 0$.

$$1] \text{ Compute } J_4 = J(\vec{v} - \frac{\rho}{4} \vec{W}), J_2 = J(\vec{v} - \frac{\rho}{2} \vec{W}),$$

$$J_1 = J(\vec{v} - \rho \vec{W})$$

$$2] \text{ If } J_4 \leq J_2 \leq J_1, \text{ do } \rho = \rho/2 \text{ and go to 1]}$$

$$3] \text{ If } J_4 > J_2 > J_1, \text{ do } \rho = 2 * \rho \text{ and go to 1]}$$

$$4] \text{ If } J_4 \leq J_2 > J_1, \text{ do } \rho = \rho/2 \text{ and go to 1]}$$

$$5] \text{ If } J_4 > J_2 \leq J_1, \rho^* \text{ is the}$$

minimum of the parabola which passes through the points

$$(\rho/4, J_4), (\rho/2, J_2), (\rho, J_1).$$

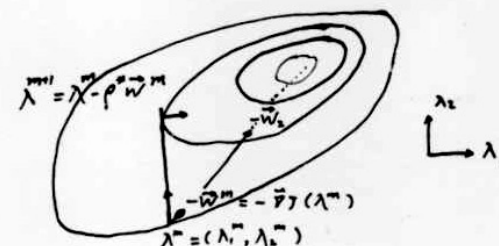
Then $(\lambda_i)_{i=1}^N = (\lambda_i)_{i=1}^N - \rho^* \vec{W}$ approaches the minimum of the function J on the line descending from $(\lambda_i)_{i=1}^N$ in the descent direction $-\vec{W}$. After computing $\vec{W} = \vec{v} J((\lambda_i)_{i=1}^N)$,

and returning to the initial state 1]...and so on.

Remarks: 1. This research can be long and require the computation of $J(\vec{v} - \rho \vec{W})$ for many points $\rho > 0$. Thus it is not possible use the code "Centaure" directly.

2. It is very important to choice a good initial $\rho > 0$, not too large or too small.

2.1 - $\vec{W} = -\vec{v} J((\lambda_i)_{i=1}^N)$ will not, most likely, be in the direction of steepest descent [on the drawing $-\vec{W}_2$ is better).



2. - An application of the gradient method on the problem of ultracentrifugation.

a] Let us assume that v_i is an element of a base $(v_i)_{i=1}^N$ of X_h , $[\hat{u}_i, \hat{v}_i, \hat{w}_i]$ the associated velocity field, solution of the state of the system [7]; $\psi_i(\xi, \eta) = \int_0^\xi E_R(\xi') \hat{w}_i(\xi', \eta) \xi' d\eta$ and $J_1(i)(\eta)$

$= 2S_c \xi^{-1} \int_0^1 \psi_i(\xi, \eta) \xi d\xi$; then the application $v_i \rightarrow J_1(i)$ is linear; thus, if $v = \sum_{i=1}^N \lambda_i v_i$ then $J_{1v}(\eta) = \sum_{i=1}^N \lambda_i J_1(i)(\eta)$ [relation 1].

Let us define $J_2(i, j)(\eta) = 2S_c^2 \xi^{-2} \int_0^1 \psi_i \psi_j(\xi, \eta) \frac{d\xi}{\xi}$.

Thus $J_{2v}(\eta) = \sum_{i, j=1}^N \lambda_i \lambda_j J_2(i, j)(\eta)$ [relation 2].

The relations 1 and 2 show that, given the functions $J_1(i)$ $i=1, N$ and $J_2(i, j)$ $i=1, N$ $j=1, N$, we easily allow the computation of J_{1v} and J_{2v} for $v = \sum_{i=1}^N \lambda_i v_i$, where $(\lambda_i)_{i=1}^N \in \mathbb{R}^N$.

b] Assume $(\lambda_i)_{i=1}^N \in \mathbb{R}^N$ and $\vec{W} = \vec{\nabla} J((\lambda_i)_{i=1}^N)$ are known, we want find a simple method to compute $J((\lambda_i)_{i=1}^N - \rho \vec{W})$ for all $\rho \in \mathbb{R}^N$.

We define $v = \sum_{i=1}^N (\lambda_i - \rho w_i) v_i$, then

$$J_{1v}(\eta) = \left[\sum_{i=1}^N \lambda_i J_1(i)(\eta) + \rho \sum_{i=1}^N w_i J_1(i)(\eta) \right] \quad (\text{relation 3}),$$

$$J_{2v}(\eta) = \left[\sum_{i, j=1}^N \lambda_i \lambda_j J_2(i, j)(\eta) \right]$$

$$+ \rho \sum_{i, j=1}^N [-2 \sum_{i, j=1}^N \lambda_i w_j J_2(i, j)(\eta)]$$

$$+ \rho^2 \sum_{i, j=1}^N w_i w_j J_2(i, j)(\eta) \quad (\text{relation 4})$$

The expressions in brackets are independent of ρ , and dependent only on $(\lambda_i)_{i=1}^N$ and \vec{W} . Then they will be computed once only, to find $\rho^* > 0$ minimizing $J((\lambda_i - \rho w_i)_{i=1}^N)$, $\rho > 0$. To compute $J((\lambda_i - \rho w_i)_{i=1}^N)$, we must compute $\alpha_s(\rho)$, that is,

$$G(\eta_F)_\rho = \int_0^{\eta_F} \frac{-\phi_w + 2\xi_0 J_1^\rho(\eta')}{1 + J_2^\rho(\eta')} d\eta',$$

$$F(H)_\rho = \int_{\eta_F}^H \frac{\phi_\rho + 2\xi_0 J_1^\rho(\eta')}{1 + J_2^\rho(\eta')} d\eta',$$

and

$$A^\rho = \int_0^{\eta_F} \frac{\exp(-G_\rho(\eta))}{1 + J_2^\rho(\eta)} d\eta,$$

$$B^\rho = \int_{\eta_F}^H \frac{\exp[-F(\eta)_\rho]}{1 + J_2^\rho(\eta)} d\eta.$$

Without going into the details, we use numerical integration so, we need the values of J_1^ρ and J_2^ρ , (according to the relations 3 and 4) at different fixed points of a partitions of $[0, \eta_F]$ and $[\eta_F, H]$. These values are computed by the Centaure code, once for all, and are independent of $(\lambda_i)_{i=1}^N$ or \vec{W} .

We use the same method to compute $\vec{\nabla} J_\rho((\lambda_i)_{i=1}^N)$,

$$w_j = \frac{\partial J((\lambda_i)_{i=1}^N)}{\partial \lambda_j} = F^* \left\{ -\theta_* \frac{\partial J_{\alpha_s}}{\partial \alpha_s} + \theta_* \frac{\partial J_{\alpha_s}}{1 + \theta(\alpha_s - 1)} \right\} \quad \text{with}$$

$$\partial_j \alpha_s = \frac{\partial \alpha_s((\lambda_i)_{i=1}^N)}{\partial \lambda_j}$$

$$\begin{aligned}
& \frac{\exp[G(\eta_F)] * \partial_j G * \left\{ 1 + \phi \int_0^{\eta_F} \frac{\exp[-G(\eta)]}{1+J2(\eta)} d\eta \right\}}{\exp[-F(H) + \phi \int_0^{\eta_F} \frac{\exp[-F(\eta)]}{1+J2(\eta)} d\eta]} \\
& + \frac{\exp[G(\eta_F)] * \phi \int_0^{\eta_F} \partial_j k(\eta) d\eta}{\exp[-F(H)] + \phi \int_0^{\eta_F} \frac{\exp[-F(\eta)]}{1+J2(\eta)} d\eta} \\
& - \frac{\exp[G(\eta_F)] \left\{ 1 + \phi \int_0^{\eta_F} \frac{\exp[-G(\eta)]}{1+J2(\eta)} d\eta \right\}}{(\exp[-F(H)] + \phi \int_0^{\eta_F} \frac{\exp[-F(\eta)]}{1+J2(\eta)} d\eta)^2} \\
& * \left\{ - \partial_j F(H) \cdot \exp[-F(H)] + \phi \int_0^{\eta_F} \partial_j M(\eta) d\eta \right\},
\end{aligned}$$

$$\begin{aligned}
\text{where } \int_0^{\eta_F} \partial_j K(\eta) d\eta &= \int_0^{\eta_F} [-\exp(G(\eta)) * \frac{\partial_j G(\eta)}{1+J2(\eta)} \\
&- \frac{\exp[-G(\eta)]}{(1+J2(\eta))^2} * \partial_j J2(\eta)] d\eta,
\end{aligned}$$

$$\begin{aligned}
\partial_j M(\eta) &= -\exp[-F(\eta)] * \partial_j F(\eta) * \frac{1}{1+J2(\eta)} \\
&- \exp[-F(\eta)] * \frac{\partial_j J2(\eta)}{(1+J2(\eta))^2},
\end{aligned}$$

$$\begin{aligned}
\partial_j G(\eta_F) &= \int_0^{\eta_F} \frac{2\xi_0 J1(j)(\eta')}{1+J2(\eta')} - \frac{(-\phi_w + 2\xi_0 J1(\eta'))}{(1+J2(\eta'))^2} \\
&* \partial_j J2(\eta') d\eta',
\end{aligned}$$

$$\begin{aligned}
\partial_j F(H) &= \int_{\eta_F}^H \frac{2\xi_0 J1(j)(\eta')}{1+J2(\eta')} - \frac{(\phi_0 + 2\xi_0 J1(\eta'))}{(1+J2(\eta'))^2} \\
&* \partial_j J2(\eta') d\eta',
\end{aligned}$$

$$\partial_j J2(\eta) = 2 * \sum_{i=1}^N \lambda_i J2(i, j)(\eta').$$

A lot of expressions have been compute before, and we use the some method to compute the others: we need only the values of and $J1(i)$ and $J2(i, j)$ $i=1, N$ $j=1, N$ at differents fixed points.

The Centaure code computes $(\hat{u}_i, \hat{v}_i, \hat{w}_i)$ for each element v_i of the base $(v_i)_{i=1}^N$, then it computes $J1(i)$ (x_e) and $J2(i, j)$ (x_e) $i=1, N$ $j=1, N$ $\{x_1, \dots, x_L\}$ a partition of $[0, \eta_F] \cup [\eta_F, H]$; that is, we use N times, and only N , Centaure code, and we spake an optimization directly with the code.

3.1 - Improvements of the gradient method.

1° Choice of the initial P.

We try three options:

a) We assume

$$J((\lambda_i)_{i=1}^N - \rho \frac{\vec{w}}{\|\vec{w}\|}) = a\rho^2 + b\rho + c$$

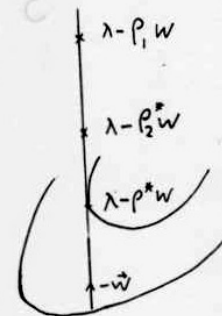
then $c = J_{\rho=0} = J((\lambda_i)_{i=1}^N)$

$$- \frac{\vec{w}}{\|\vec{w}\|} = \frac{\partial J}{\partial \rho}(\rho=0) = (2a\rho + b)_{\rho=0} = b$$

so $b = - \frac{\vec{w}}{\|\vec{w}\|}$

We assume also $J((\lambda_i)_{i=1}^N) - \xi \leq J((\lambda_i)_{i=1}^N)$

$$- \rho * \frac{\vec{w}}{\|\vec{w}\|} \leq J((\lambda_i)_{i=1}^N)$$



Consequently $-\xi \leq a\rho^2 - \|w\|\rho \leq 0$
: thus the

minimum ρ^* verify $\rho^* = -\frac{b}{2a} = \frac{\|w\|}{2a}$,

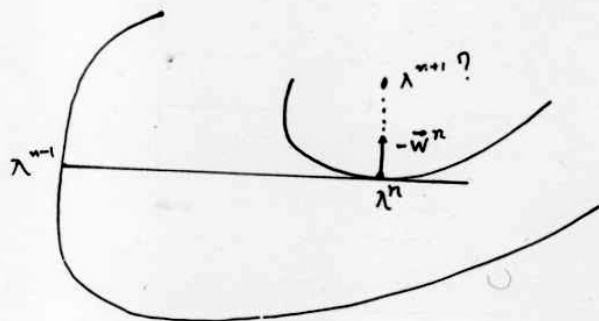
so $-\xi \leq \frac{1}{a} \frac{\|w\|^2}{4} : a \geq \frac{\|w\|^2}{4\xi}$,

we assume $\xi \leq \frac{5}{100}$ then $a \geq 5\|w\|^2$; the largest p initial is $\rho_{init} = \frac{1}{10\|w\|}$ (associated with $a = 5\|w\|^2$).

b) Yet, the choice of $\xi = \frac{5}{100}$ is not the best.

Assume $\xi = \Delta J$ then p initial $= \frac{2\Delta J}{\|w\|^2}$. After a computation we have verified that p initial $= \frac{2(J_n - J_{n+1})}{\|w\|^2}$ is very good, but if we use this

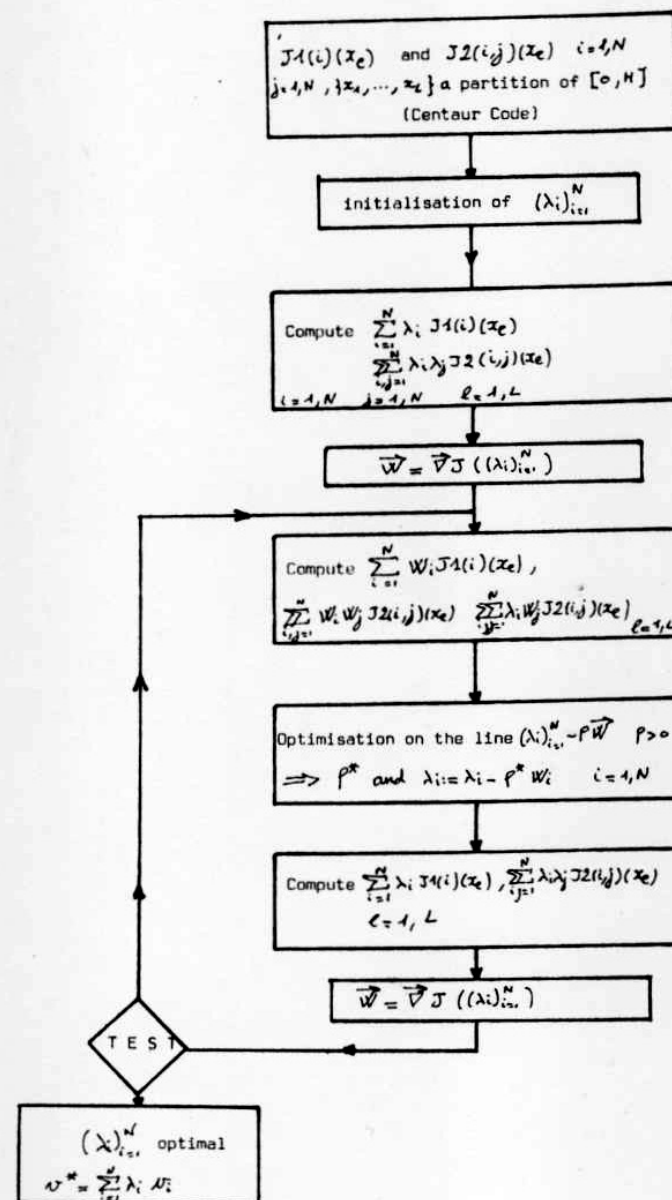
choice, with $\Delta J = J_{n-1} - J_n$ (as λ^{n+1} , thus $J_{n+1} = J(\lambda^{n+1})$ are unknown), the result is different because at each change of direction \vec{w} the variation ΔJ defers.



c) So we must find a better evaluation of ΔJ .

We assume $\frac{\Delta J_2}{\Delta J_1} = \frac{\|w_2\|}{\|w_1\|}$ (linear hypothesis).

2. - DIAGRAM.



In particular, this choice is better during the first iterations (half of iterations). For that, we establish that the direction $-\vec{w} = -\vec{\nabla}J((\lambda_i)_{i=1}^N)$ is not very good. We try several algorithms to find a faster method for our problem.

3.2 - Elliptic approximation.

We consider that the isovalues of $J((\lambda_i)_{i=1}^N)$, around $\lambda = (\lambda_i)_{i=1}^N \in \mathbb{R}^N$, are ellipses of which we search the

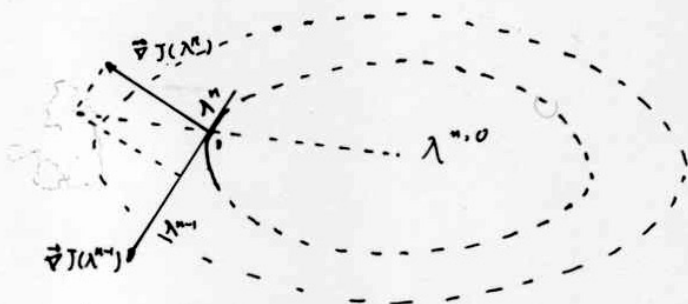
center λ^0 : λ^0 will be the descent direction.

Precisely, we know λ^{n-1} , $J(\lambda^{n-1})$, $\vec{\nabla}J(\lambda^{n-1})$, λ^n , $J(\lambda^n)$, $\vec{\nabla}J(\lambda^n)$ and we search for θ_n such that

$\vec{w}^n = \vec{\nabla}J(\lambda^n) + \theta_n \vec{\nabla}J(\lambda^{n-1})$ verifies $-\vec{w}^n = \alpha \lambda^0$, $\lambda^{n,0}$,

where $\alpha > 0$ and $\lambda^{n,0}$ is the center of the approximate ellipses.

Notation: $\rho^{n-1} = ||\lambda^n - \lambda^{n-1}||$.



We assume that the ellipses passing through λ^{n-1} and λ^n have the same ratio of axe lenght, the some axes and the some center. Then we can compute the center $\lambda^{n,0} \in \mathbb{R}^N$:

$$\lambda_i^{n,0} = \frac{\rho^{n-1} \partial_i J(\lambda^n) \lambda_i^{n-1} - (\lambda_i^{n-1} - \lambda_i^n) \lambda_i^n}{\rho^{n-1} \partial_i J(\lambda^n) - (\lambda_i^{n-1} - \lambda_i^n)} \quad i = 1, N$$

and after

$$\theta_n = \frac{(\lambda^n - \lambda^{n,0}) \cdot \vec{\nabla}J(\lambda^{n-1})}{(\lambda^n - \lambda^{n,0}) \cdot \vec{\nabla}J(\lambda^n)}.$$

Then $\vec{w} = \vec{\nabla}J(\lambda^n) + \theta_n \vec{\nabla}J(\lambda^{n-1})$ verifies the relation wanted.

Remarks: - The approximation is only around λ^n , depending on λ^{n-1} and λ^n , and changes at each iteration.

- The convergence is fast, especially at the start, and the descent is very continuous.

III. Results and Discussion

We present an example of calculation for a centrifuge specified by the data

Height 250 cm ,

Radius 25 cm ,

Rotation speed parameter $A^2 = M\omega^2 a^2 / 2 RT_0 = 25$.

Ekman number at the periphery $\epsilon = \mu / \rho_w \omega a^2 = 6.5 \cdot 10^{-8}$,

Feed (at midheight of the centrifuge) $F = 0.1$ g

UF₆/S

Cut $\theta = 0.5$.

The data $A^2 = 25$ and $\epsilon = 6.5 \cdot 10^{-8}$ were calculated with the following values of physical parameters:

Gas pressure at the periphery $P_w = 100$ torrs ,

Mean gas temperature $T_0 = 310$ K ,

Peripheral speed $\omega a = 605$ m/s .

As first step of computations, we have partitionned the

boundary $\delta\Omega$ into 19 intervals (listed in TABLE 1) and we have considered 19 base functions piecewise linear and continuous. The next step is to compute the velocity field corresponding to each component of the basis with the help of CENTAURE code and to constitute a file of 19 velocity fields. The final step is to proceed to the optimization calculation by the method of Sec II. However one should keep in mind the fundamental assumption underlying the model, i.e. the gas flow is assumed to be a small perturbation around the rigid body rotation. The linearization of the hydrodynamics equations is valid only for small ROSBY numbers. In consequence, the optimal temperature profile on the boundary that we are looking for must verify everywhere on the boundary the constraint

$$\left| \frac{T - T_0}{T_0} \right| \ll 1 ,$$

we have computed three cases

$$(a) \quad |(T - T_0)/T_0| \leq 5\% ,$$

$$(b) \quad |(T - T_0)/T_0| \leq 10\% ,$$

$$(c) \quad \text{No constraint on } |(T - T_0)/T_0| .$$

The three corresponding optimal Controls, i.e. the optimal profiles of the temperature on the solid boundaries, are plotted on Fig. 1. A certain number of comments are to be made upon the results of these calculations:

- 1 - In all the three cases, the optimal drive is predominantly the end caps drive and the lateral

mode is practically inexistent.

- 2 - The optimal separative power depends strongly on the value of the constraint in the range 0 to 10% of ROSBY number; but it does not change much beyond the value 10% for the ROSBY number. In the case (c) (no constraint), the peak of the temperature profile is high (20%). The validity of the linearization seems doubtful.
- 3 - The same centrifuge has been optimized in Ref[1] by a simplified approach. The separative power obtained in [1] was 38 SWU/year, i.e a little bit lower than the value 41.3 SWU/year obtained by the present general method. Nevertheless it should be pointed out that the control obtained by the simplified method is much easier to realize in practice than the sophisticated temperature profile of Fig. 1 yielded by the present work.
- 4 - The mass velocity field in the optimal situation is plotted in Fig. 2. Instead of two loops of the standard picture of countercurrent flow, the optimal flow is structured into four loops.
- 5 - The temperature profile in Fig. 1 is not exactly antisymmetric. We have also proceeded to the optimization calculation by imposing the condition of antisymmetry as a constraint. The temperature profile obtained in that case is plotted in Fig. 3 and the corresponding optimal separative power is 40.8 SWU/year, i.e slightly lower than in case (c) of Fig. 1. Moreover, though the profile of Fig. 3 is qualitatively similar to that of Fig. 1, there is a very big difference in the peak values of the temperature profiles: 28% in Fig. 3 instead of 20% in Fig. 1.

6 - In order to study the effect of the gas pressure on the optimal performance of a centrifuge, we have carried out computations with the same data of centrifuge as above except for the gas pressure at the periphery, for which we have considered two more cases

$$P_w = 400 \text{ torrs and } P_w = 20 \text{ torrs.}$$

For sake of simplicity, the computations are done by assuming the antisymmetry. The optimal distributions of temperatures on the solid boundaries for the three cases are plotted in Fig. 4. The optimal separative powers and the peak values of the temperature profiles for the three cases are as follow:

P_w torrs	ΔU optimal SWU/year	$ (T-T_0)/T_0 $	comment
20	34.	68.3%...	} validity of linearization doubtful.
100	40.8	27.6%...	
400	40.1	7.5%	

For decreasing values of the pressure, the optimal temperatures are increasing, which is quite normal because the separative performance depends upon the mass flow rate of the countercurrent. Lower pressure results in lower quantity of gas in the centrifuge and requires then higher values of temperature to activate the countercurrent. The dependency of the optimal separative power upon the pressure is more striking. It seems that the performance gets saturated, as it can be seen from

Fig. 5, while the BROUWER'S estimation [10] gives an increasing efficiency with increasing pressure. This point needs further investigation.

CONCLUSION.

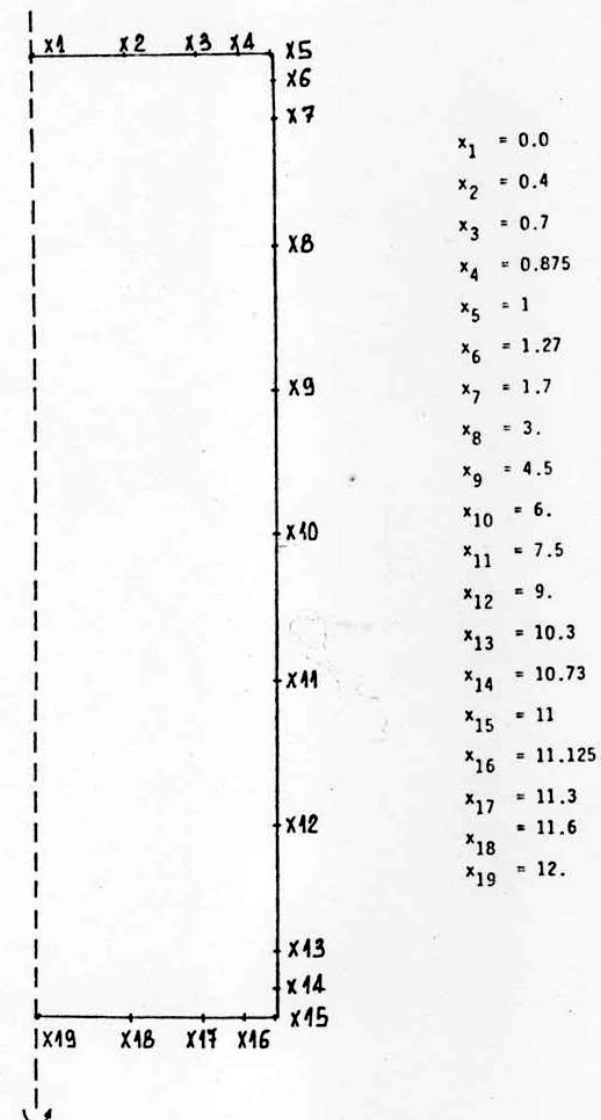
The method of optimization of a centrifuge developed in this paper is more general than what we have done previously. Even though the optimal boundary conditions obtained by the present method seem too sophisticated to be realizable in practice, the computation is nevertheless quite instructive for at least two reasons:

- 1 - It yields an upper limit of the performance that can be expected from a specified centrifuge by the internal countercurrent mechanism.
- 2 - It provides a guideline to select practically boundary conditions, i.e. as close as possible to the optimal profiles derived by the present method.

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TABLE: 1*Partition of the boundary*

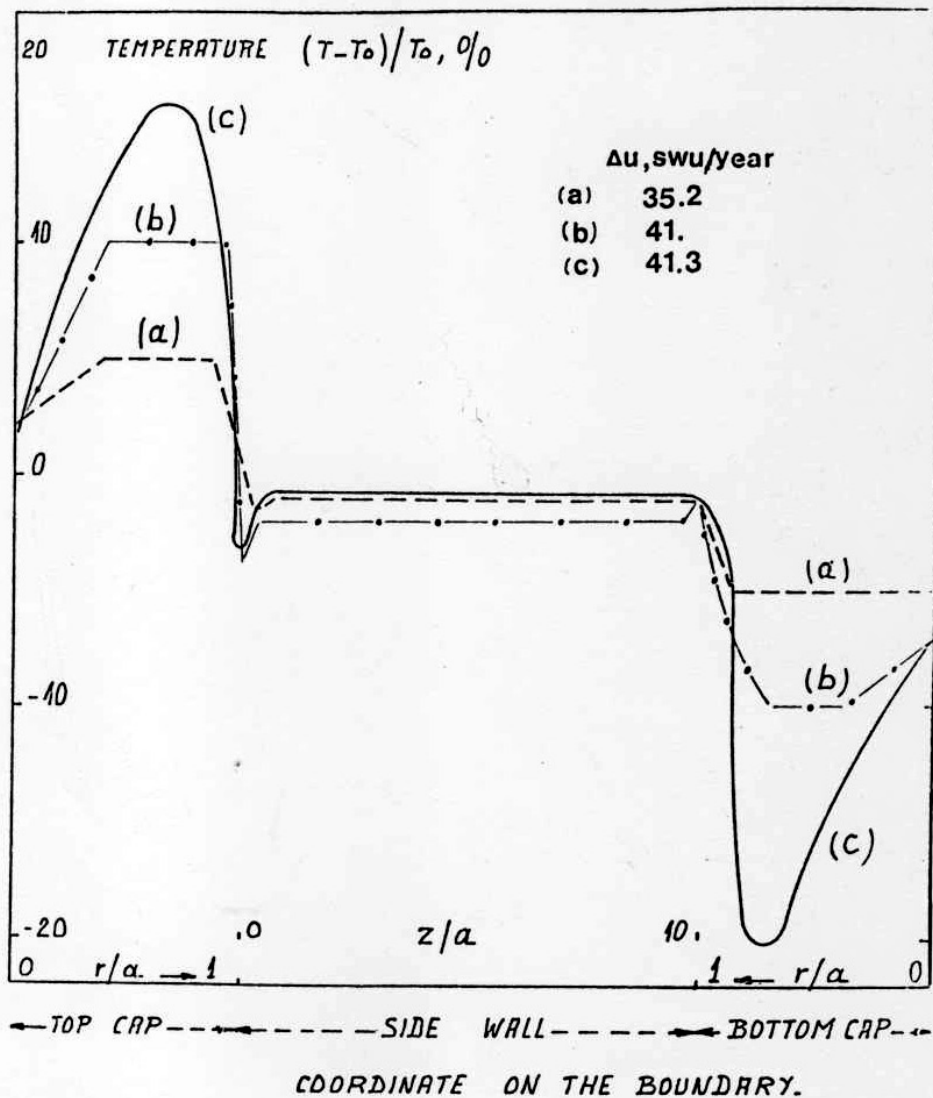


Fig. 1 Optimal temperature distribution on the solid boundaries of a centrifuge satisfying the constraint $|(T - T_0)/T_0| \leq \delta_0$. Three cases are considered:
(a) $\delta_0 = 5\%$; (b) $\delta_0 = 10\%$; (c) no limitation on δ_0 .

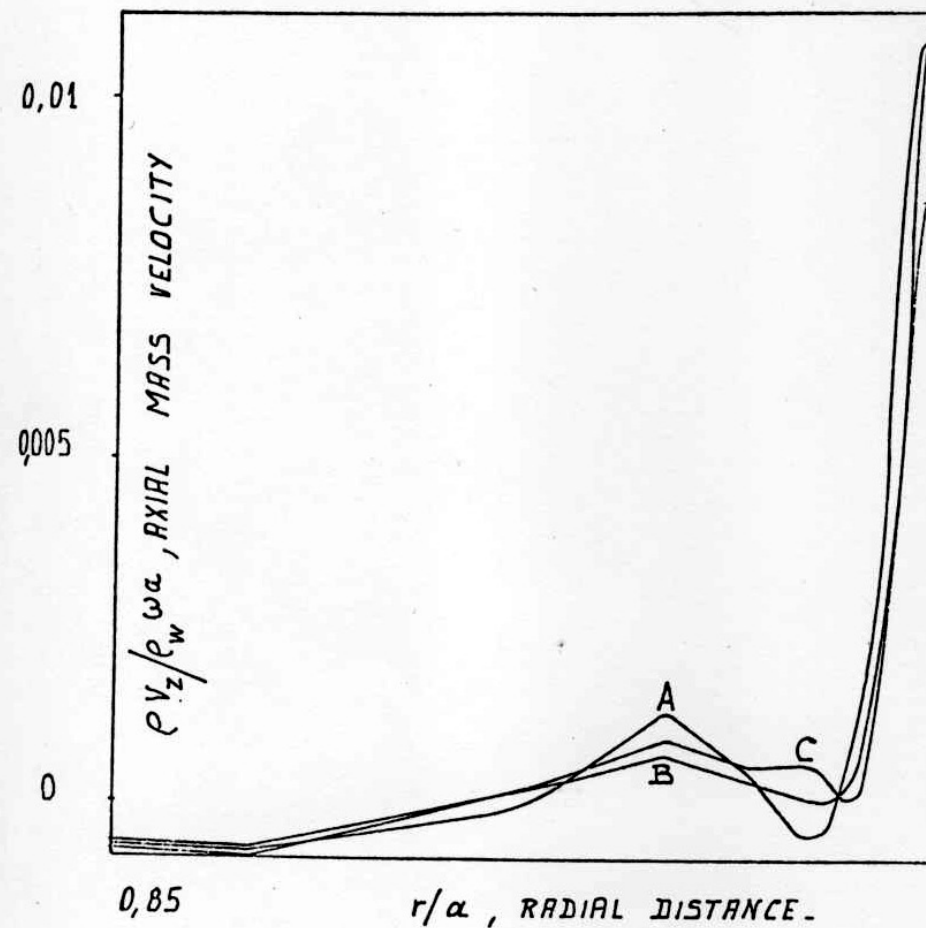


Fig. 2 Axial mass velocity versus radial coordinate in the optimal situation, at three cross sections
A : $z/h = 0.5$; B : $z/h = 0.75$; C : $z/h = 0.25$

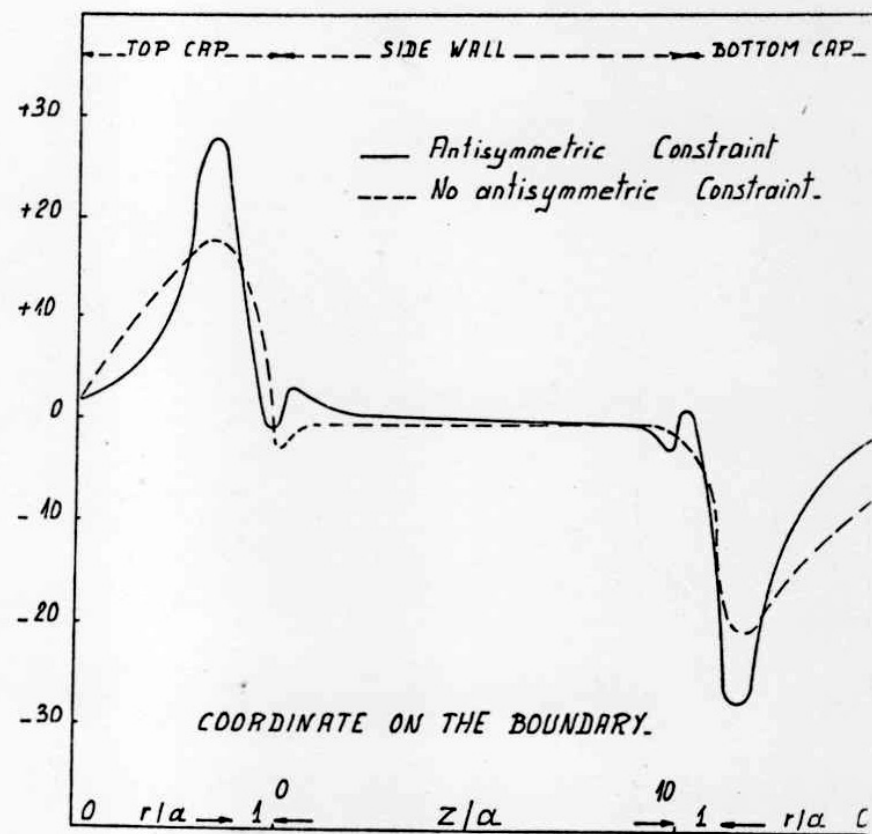


Fig. 3 Optimal temperature distribution on the solid boundaries, the profile being constrained to be antisymmetric. On the same figure is also plotted the solution without antisymmetric constraint.

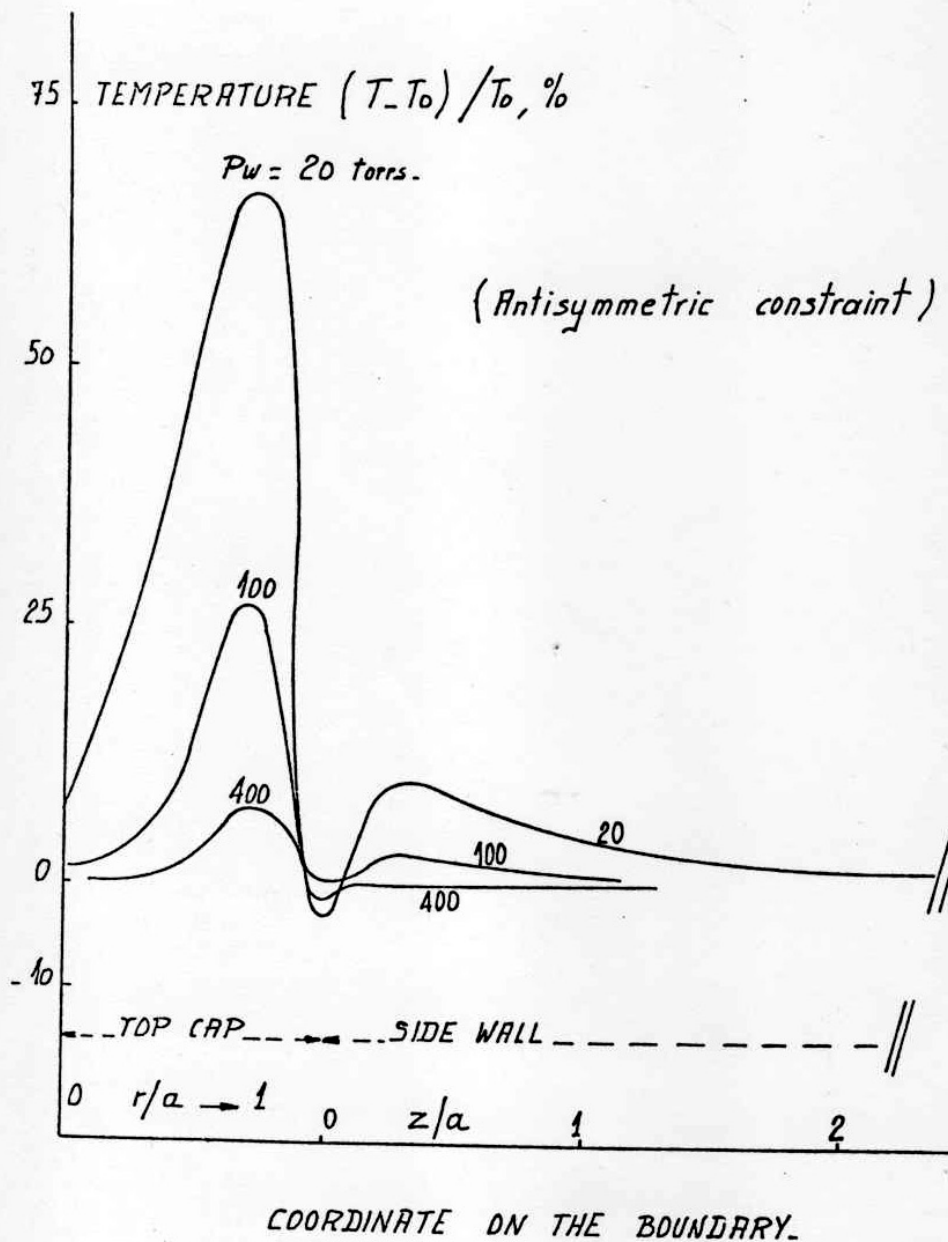


Fig. 4 Influence of the gas pressure p_w at the periphery on the optimal profile of temperature on boundaries.

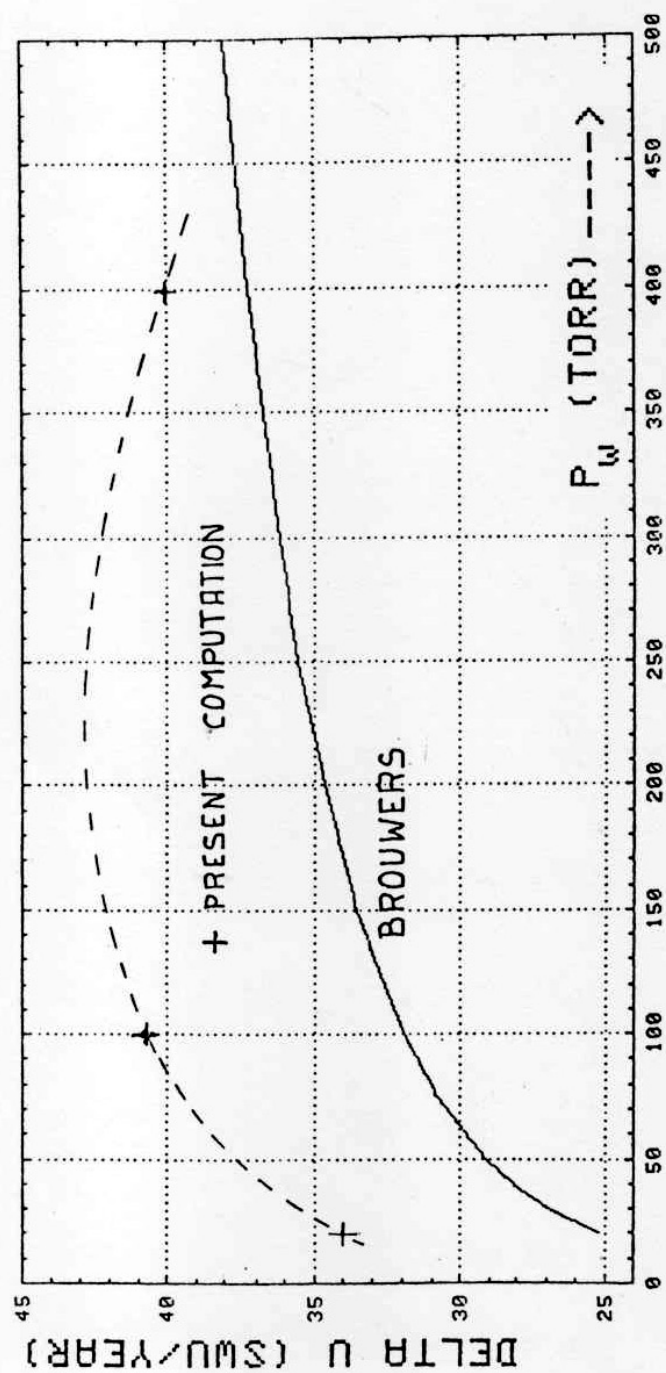


Fig. 5 Dependency of the optimal separative power upon the gas pressure at the periphery.